Rough Sketches

Sorry, I ran out of time, so I could only give rough sketches of the solutions.

Problem 11.7

Drop the perpendicular from C to AB and let the base be D. We have $\angle ADC = \angle ACB$ and $\angle DAC = \angle CAB$, so $\Delta ADC \sim \Delta ACB$. So we get $\frac{AD}{AC} = \frac{AC}{AB}$, or $AD = b^2/c$. Similarly we get $BD = a^2/c$. But $AD + BD = AB$.

Problem 11.7.2.10

For existence, if s is 1 or prime then we're done. Otherwise $s = ab$ for some $a, b < s$, so induct to get $a = p_1 \dots p_m$ and $b = q_1 \dots q_n$, so $s = p_1 \dots p_m q_1 \dots q_n$. For uniqueness, if $s = p_1 \dots p_m = q_1 \dots q_n$, by Euclid's lemma p_1 divides some q_i; without loss of generality it's q_1. Then $p_1 = q_1$, so induct on s/p_1.

Problem 11.5

Consider pairs of (e, v) where e is an edge and v is an endpoint of e. Each edge is counted twice, so the number of pairs is even. However, each v is counted $\deg v$ times. So the sum of all degrees is even; if there were an odd number of odd vertices the sum would be odd.

Problem 6.7.10

Consider the quadratic $\sum(u_i x + v_i)^2$. It has at most one real root, so its determinant $4(\sum u_i v_i)^2 - 4(\sum u_i^2)(\sum v_i^2)$ is non-positive.

Problem 12.5.7

Without loss of generality suppose $f(a) < u < f(b)$. Define S as all $x \in [a, b]$ where $f(x) \leq u$. S is nonempty because $a \in S$ but is bounded above by b, so $c = \sup S$ exists. Let $\epsilon > 0$; there exists $\delta > 0$ such that whenever $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$. Since c is supremum of S, there exists $a^* \in (c - \delta, c] \cap S$, so $f(c) < f(a^*) + \epsilon \leq u + \epsilon$. However, for all $b^* \in (c, c + \delta)$, $b^* \notin S$, so $f(c) > f(b^*) - \epsilon > u - \epsilon$. Now take $\epsilon \to 0$.

Problem 5.8.7

Let (a_i, b_i) be the lengths of longest increasing/decreasing subsequence ending on x_i respectively. Then no pair is repeated. There aren't enough pairs in $[1, r-1] \times [1, s-1]$, so there is some pair with $a_i \geq r$ or $b_i \geq s$.

Problem 3.8.7

For each unit vector p, define $\pi(p)$ as the half-space that is normal to p, in direction of p, and contains exactly half of A_n; if there are multiple such spaces, take the middle one. Define $f(p) = \big(\mathrm{vol}(A_1 \cap \pi(p)), \dots, \mathrm{vol}(A_{n-1} \cap \pi(p))\big)$. Then f is continuous; by Borsuk–Ulam theorem, there exists p where $f(-p) = f(p)$.

Problem 13.10

It suffices to prove the upper bound. Let σ be such that $\sum_i x_{\sigma(i)} y_i$ is maximum; if there are many such σ, take one with the most fixed points. Suppose σ is not the identity, then there exists a smallest non-fixed point j of σ, and a $k > j$ where $\sigma(k) = j$. Then the permutation σ' obtained by swapping j and k will either give a larger sum or fix more elements.

Problem 2.6.5.7

Let X be the space of all labelings of G with k colors, and give it the product topology $k^{V(G)}$. By Tychonoff's theorem X is compact. For each finite subgraph H, let X_H be the subset of X that colors H properly. Then X_H is closed and the family of all X_H's have finite intersection property; by compactness they have nonempty intersection. Let c be a member in the intersection. Since $c \in X_e$ for each edge e of G, c colors G properly.

Problem 9.5

For each x, define $\alpha(x) = \min_{S \in \mathcal{F}, x \notin S} w(S) - \min_{S \in \mathcal{F}, x \in S} w(S \setminus \{x\})$. Then $\alpha(x)$ doesn't depend on $w(x)$. As $w(x)$ is taken uniformly from $\{1, 2, \dots, N\}$, we get $\alpha(x) = w(x)$ with probability $\leq 1/N$. So the probability $\alpha(x) = w(x)$ for some x is $\leq n/N$. Now, if there are two sets $A, B \in \mathcal{F}$ with the same minimum weight, then for any $x \in A \setminus B$, we have the equality $\alpha(x) = w(B) - \big(w(A) - w(x)\big) = w(x)$ that only happens with probability $\leq n/N$.

Problem 5.1.7

The system $\sum_{i=1}^{d+2} a_i x_i = 0$ and $\sum_{i=1}^{d+2} a_i = 0$ has $d + 1$ equations (one for each dimension d, plus the last equation) but has $d + 2$ unknowns, so there exists a solution a_i's not all zero. Let X, Y be the set of indices where a_i is ≥ 0 and < 0 respectively. Then we have $\sum_{i \in X} \frac{a_i}{A} x_i = \sum_{i \in Y} \frac{-a_i}{A} x_i$ where $A = \sum_{i \in I} a_i$, and each side is a convex combination of points $\{x_i\}_{i \in X}$ and $\{x_i\}_{i \in Y}$, so this is a point in the convex hulls of $\{x_i\}_{i \in X}$ and $\{x_i\}_{i \in Y}$.

Bonus problem

By Definitions 6.5, 1.1, 5.4, 8.2, 2.4, 10.3, 2.1, 7.4, 1.7, 4.3, 3.3, 6.3, 5.1, 4.7, 9.3, 10.4, 8.3, 11.5, 3.8, 11.4, 7.6, 9.2, we get the answer.